Projectile Motion

Reference Guide

Equipment Setup

Investigation Guides

C-1 Projectile Motion and the Range Equation

Key Question: How can you predict the range of a launched marble?

In this Investigation, students derive an equation that allows them to predict the range of the marble given the initial velocity and launch angle. First, they derive the range equation. Next, they test their predictions. Finally, they compare theoretical predictions to actual measurements using a range vs. launch angle graph. Stress safety when using the marble launcher.

$$x = \frac{2v^2 \sin\theta \cos\theta}{g}$$

Preparation

This Investigation assumes that students are already familiar with the concept of projectile motion. Students may need to do the first two Level B Investigations prior to this one. In addition, you may need to review trigonometry and vectors.

The Investigation begins with a derivation of the range equation from consideration of the x and y components of the marble's initial velocity. It is assumed that the marble starts and ends at the same elevation (y = 0). The resulting range equation is used to make predictions about the motion of the marble which are tested by experiment.

Setup and Materials

Students work in groups of four at tables.

Each group should have:

- Marble launcher
- At least one black plastic marble per group. At least one spare marble per group is recommended.
- Ruler, meter stick, or tape measure
- Scientific calculator
- Roll of masking tape

Each student should have:

- Copy of the Investigation and answer sheet

The Investigation

Time

One class period

Leading Questions

- What is projectile motion?
- How can you derive the range equation?
- How does theory compare to actual measurements?

Learning Goals

In this Investigation, students will:

- Derive the range equation.
- Test the theory with actual measurements.
- Compare predictions to measurements.

Key Vocabulary

projectile motion, trajectory, range equation

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1a. The path of the marble is a curved path because after the initial launch, the marble is acted upon by the force of gravity. The force of gravity pulls down, while the initial thrust pushes horizontally.

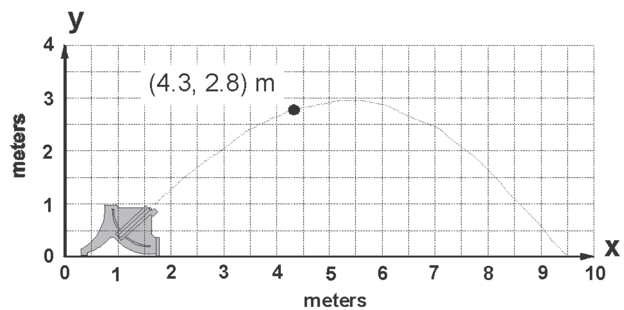
1b. The marble's velocity in x stays the same during the flight time. The marble's velocity in y decreases over time.

Teaching Notes: Cartesian (x,y) Coordinates

The motion of the marble follows a curved path through space. To tell where the marble is at any moment, a method must be devised to measure space in two directions. One method is called the **Cartesian**, or **x-y coordinate system**.

In the Cartesian system, there are two axes, the x and the y that are constructed perpendicular to each other as shown below. The position of the marble at any time can be specified by the coordinates. The coordinates are specified as a pair of measurements (x, y) where x and y are distances along the x and y axis. For example, the coordinates of the marble in the figure below are (4.3, 2.8) meters.

With Cartesian coordinates, the position is specified relative to the origin. The origin is the point with coordinates $(x, y) = (0, 0)$. The location of the origin is usually chosen to be convenient for the experiment.



C-1 Projectile Motion and the Range Equation

C-1

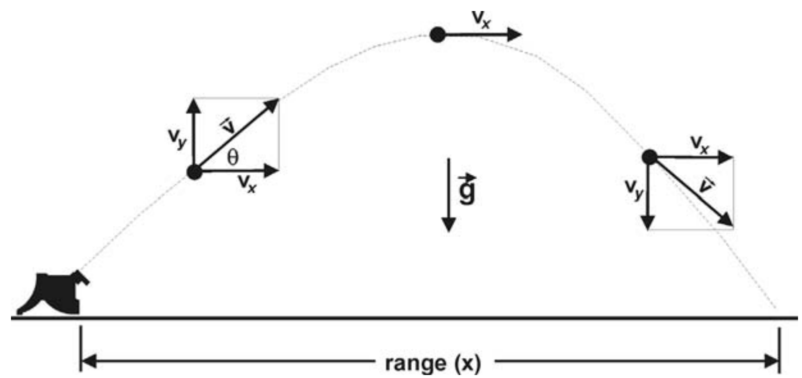
Question: How can you predict the range of a launched marble?

In this Investigation, you will find and test a model that will predict the range of the marble from the initial velocity and launch angle.

You have learned that the motion of any object moving through the air affected only by gravity is an example of **projectile motion**. Examples of projectile motion include a basketball thrown toward a hoop, a car driven off a cliff by a stunt person, and a marble launched from the CPO marble launcher. Projectile motion is also called two-dimensional motion because it depends on two components: vertical and horizontal. In this Investigation, you will determine a mathematical model (the **range equation**) that predicts the range of the marble given launch angle and initial velocity.

1 Analyzing the motion of the marble in two dimensions

How can you predict the range of the marble? Since gravity pulls *down* and not sideways, the motion of the marble must be separated into *components*. It makes sense to pick one component (y) in the vertical direction aligned with gravity. The other component (x) is then chosen to be in the horizontal direction, *perpendicular* to the force of gravity. The diagram below shows the velocity of the marble (v) at three points in its trajectory, resolved into x and y components, v_x and v_y.



a. Use the diagram above to explain why projectiles travel in a curved path called a *trajectory*.

b. How does the marble's velocity in x change over the time of the flight? How does its velocity in y change over the time of flight?

2 Understanding the velocity equations

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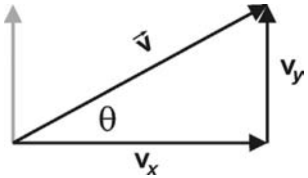
The object of this Investigation is to find and test a model that will predict the range of the marble from the initial velocity and launch angle—the range equation.

a. The first step is to separate the velocity of the marble into x and y components. Use the triangle formed by velocities (at right) to express v_x and v_y in terms of the initial velocity, v, and the sine and cosine of the launch angle, θ.



When the initial velocity is separated into x and y components, Equations 1a-2b give the relationships between the motion variables *separately* for x and y. In these equations the subscript i refers to the *initial* values at launch.

Equations 1a and 1b are for the marble's velocity while equations 2a and 2b are for the marble's position.

Equation 1a $v_x = v_{xi} + a_x t$

Equation 1b $v_y = v_{yi} + a_y t$

Equation 2a $x = x_i + v_{xi} t + \frac{1}{2} a_x t^2$

Equation 2b $y = y_i + v_{yi} t + \frac{1}{2} a_y t^2$

b. Since gravity does not pull up or sideways, one of the accelerations (a_x, a_y) is -g, and the other is zero. For the first approximation, the range (x) is defined so the initial x and y positions are also zero. Rewrite Equations 1a-2b leaving out terms that are zero and substituting your previous results for v_{xi} and v_{yi}.

Equation 1a:

Equation 1b:

Equation 2a:

Equation 2b:

c. The purpose of this exercise is to find a theory that predicts where the marble will land (the range, x) given the initial velocity and launch angle. This problem can be solved in several steps. First, assume that gravity acts only on the y component of velocity. Solve for the *time* it takes the marble to reach its maximum height (where $v_y = 0$).

d. Since gravity does not pull sideways, the x component of the marble's velocity is not affected and remains constant. Use Equation 2a to calculate the range (x) from v_{xi} and the total time of flight (this will be the range equation).

2

2

2a. $v_x = v\cos\theta$

$v_y = v\sin\theta$

2b. Equation 1a: $v_x = v\cos\theta$

Equation 1b: $v_y = v\sin\theta - gt$

Equation 2a: $x = vt\cos\theta$

Equation 2b: $y = vt\sin\theta - \frac{1}{2}gt$

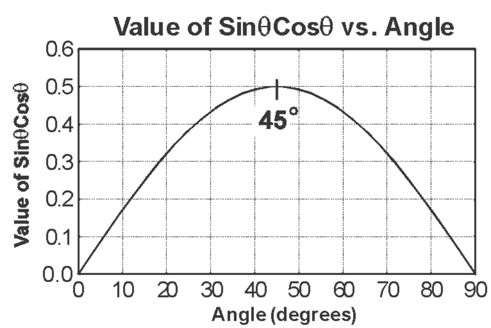
2c. $t_{top} = \frac{v\sin\theta}{g}$

2d. $x = \frac{2v^2\sin\theta\cos\theta}{g}$

Teaching Notes: The Range Equation

The range equation (**2d** above) predicts the range of the marble from two measured quantities: initial velocity (v_i) and launch angle (θ). This equation has two properties that are readily observable in experiments:

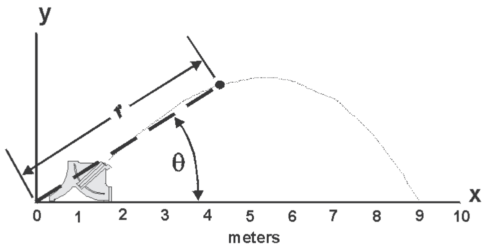
- The range increases as the square of the initial velocity.
- A graph of the product of sinθcosθ vs. launch angle shows that the range has a maximum at 45 degrees as shown below.



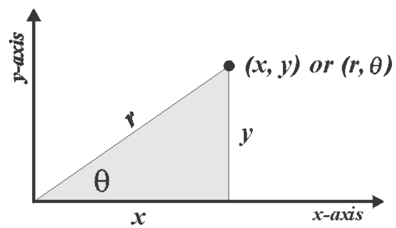
3 Student responses are not required for Part 3.

Teaching Notes: Polar (r, θ) Coordinates and Trigonometry

Another way to locate a point in two dimensions is with **polar coordinates**. Polar coordinates use the distance from the origin (radius, *r*), and the angle (θ) between the x axis and a line from the origin to the point. The figure below shows how to specify the location of the marble using polar coordinates.



In the Investigation, students convert back and forth between polar and Cartesian coordinates. This conversion is done with trigonometry, using the relationship between the three angles and three sides of a right triangle. The right triangle has a 90 degree angle. This is useful because 90 degrees is also the angle between the x and y axes of a graph. The figure below shows a right triangle with one angle and three sides labeled. The Cartesian coordinates correspond to the lengths of the x and y sides of the triangle.



In a right triangle, all of the sides and angles can be deduced knowing only two sides, or one side and one angle because the ratios of the sides depend only on the included angle. These ratios have been given the names sine, cosine, and tangent, and are:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

3 Setting up the marble launcher

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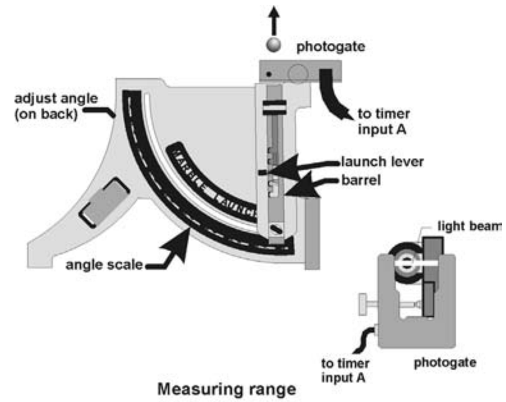
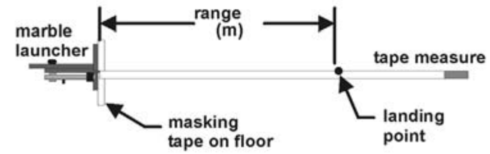
1. Attach one photogate to the marble launcher so that the marble breaks the light beam as it comes out of the barrel. Put the timer in interval mode, and connect the photogate to input A. The launcher can launch at angles from 0 (horizontal) to 90 degrees (vertical). For this experiment, you will use angles from 0 to 80 degrees.

   The photogate attaches to the tab on the end of the wood piece that supports the barrel. Be sure the light beam crosses the center of the barrel.

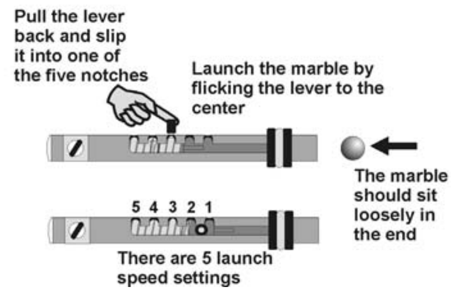


2. Use a strip of masking tape on the floor to make sure that the marble launcher is consistently placed in the same location. A tape measure laid along the floor provides a good range reference.

Measuring range



3. To launch marbles, pull the launching lever back and slip it sideways into one of the slots. Put a marble in the end of the barrel and the marble launcher is ready to launch.
4. There are five notches that change the compression on the spring and give different launch speeds. In this experiment, you will change the launch speed setting for different launches.



SAFETY RULES:

- **Never launch marbles at people.**
- **Wear safety glasses or other eye protection when launching marbles.**
- **Launch only the black plastic marbles that come with the marble launcher.**

3

4 Doing the experiment

C-1

1. To check your theory, a spread of data for different *launch angles* and *initial velocities* will be needed. For each spring setting you should have five different launch angles from 20 degrees to 80 degrees. To cover both variables (angle and speed), at least 20 data points are needed.
2. A minimum of two people are needed per launcher. One person launches the marbles and the other person/people watches where they land.
3. Use *only* the black plastic marbles provided by CPO.
4. Record the spring setting, launch angle, time from photogate A, and measured range for each launch. It often takes several launches with the same setup to locate the landing point precisely. For each setup, you may need to run several trials until the measured range is consistent within 5 cm.
5. Calculate the initial velocity by dividing the distance traveled (width of marble = 0.019 m) by the time at photogate A. Record initial velocities in the table. *A larger version of the table is found on your answer sheet.*

Spring setting (1 - 5):

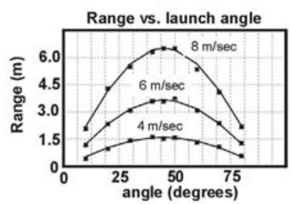
Launch angle (degrees)	Range (m)	Distance (m)	Time from A (sec)	Initial velocity (m/sec)
		0.019		
		0.019		
		0.019		
		0.019		
		0.019		

5 Comparing theory predictions to measured data

Use the table to help you compare your measured data to theory predictions using your range equation. Fill in several launch angles for each initial velocity. *A larger table is found on your answer sheet.*

Initial velocity (m/sec)		Launch angles (degrees)					
	x (predicted)						
	x (measured)						

a. Make a graph showing the range vs. launch angle for several different initial velocities. The graph at right is one example of how this graph could look. You could also choose other ways to graph the data. Plot the measured points as unconnected dots and the theoretical values as solid lines since the theory predicts the speed of the marble at all points.



b. How does your theory compare with your measurements? In particular, is there a consistent deviation between theory and experiment? The word "consistent" means the difference between the theoretical and experimental data seems to depend on something in the experiment and is not random. For example, a consistent deviation would occur if the measured range for small angles is *always* smaller than predicted by theory *regardless of the velocity*. Explain how the consistent deviations you found are affected by velocity and angle.

c. Consistent deviations indicate that something is missing from the theory. What is missing, and why does it have the observed effect on your results?

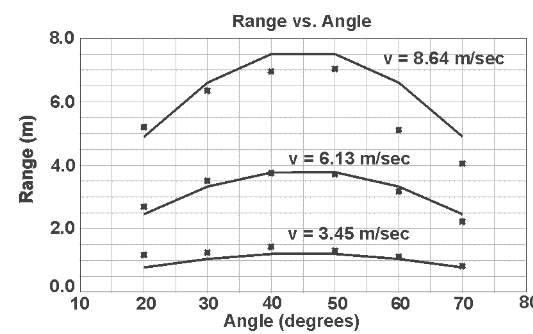
4 Sample data for spring setting 3:

Angle	Range	Time	Velocity
20	2.69	.0031	6.13
30	3.51	.0031	6.33
40	3.76	.0031	6.33
50	3.72	.0031	6.33
60	3.18	.0031	6.33
70	2.23	.0031	6.33

5 Sample data:

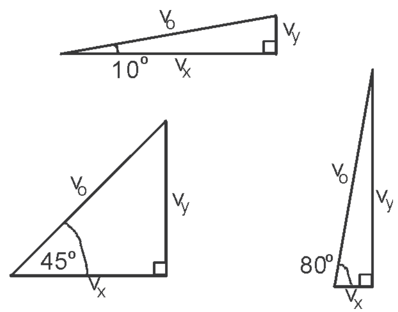
Initial velocity		Launch angles					
		20	30	40	50	60	70
3.45	*x* (pred)	0.78	1.05	1.20	1.20	1.05	0.78
	x (meas)	1.17	1.25	1.43	1.31	1.12	0.83
4.75	*x* (pred)	1.48	1.99	2.27	2.27	1.99	1.48
	x (meas)	1.96	2.31	2.39	2.33	1.48	1.42
6.13	*x* (pred)	2.46	3.32	3.77	3.78	3.32	2.46
	x (meas)	2.69	3.51	3.76	3.72	3.18	2.23
7.31	*x* (pred)	3.50	4.72	5.37	5.37	4.72	3.50
	x (meas)	4.08	5.00	5.33	5.25	4.45	3.33
8.64	*x* (pred)	4.89	6.59	7.50	7.50	6.59	4.89
	x (meas)	5.20	6.35	6.95	7.03	5.11	4.05

5a. Graph:

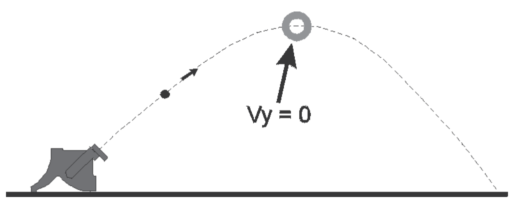


Part 5 answers continued on the next page.

1. The flight off the table will go higher and farther. The portion of the flight from the launcher to the apex of the flight (top of the arc) will be unchanged, except for starting at the height of the table. Because of this extra height, it will fly higher than the marble launched from the floor. The portion of the flight from the apex to the ground will now take longer, since it has farther to fall. Because of the longer time, it will have a longer range (i.e., the horizontal velocity is unchanged, but the time spent traveling at this velocity is no longer).
2. Answers are:



3. Answer:



4.

$$x = \frac{2(6.33)^2 \sin 80^\circ \cos 80^\circ}{9.8} = 1.40\,\text{m}$$

5 Answers, continued.

5b. There are consistent differences between theoretical and measured data. The measured value for range is consistently higher than the predicted values at small angles. The deviation becomes more noticeable at slower initial velocities.

The range is smaller than predicted for faster initial velocities at large angles.

5c. The derivation of the range equation assumed that the marble took the same amount of time to fall down as it did to go up. The real time to fall back to the ground is longer, so the range should be greater in the measured values.

The differences at faster initial velocities and large angles could be due to air friction which slows the marble down slightly.

Curriculum Resource Guide: Marble Launcher

Credits

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